Lecture 1

Sets, relations, functions. Equivalence relation.

Def. A set is a prime object in mathematics

A set is a collection of different things. These things are called elements (or members) of the set.

Examples: numbers, symbols, points in space, variables, functions, or even other sets.

A set may be specified by:

- listing its elements

$$A = \{a_1, a_2, \dots, a_n\}$$

- or by giving the first element and specifying the rule how to define the next element

$$A = \{a_1 = 0,\ a_n = a_{n-1} + 4,\ n = 2, 3, \dots\}$$

$$A = \{a_1 = 1,\ a_2 = 1,\ a_n = a_{n-1} + a_{n-2},\ n \geq 3\}.$$

- by giving a property that characterizes its elements

$$A = \{x \mid P(x)\}$$

- If x is an element of a set S, we say x belongs to S or x is in S, and this is written as $x \in S$.
- statement "y is not in S" is written as $y \notin S$.

Example. $\mathbb{Z}$ is the set of the integers one has $6 \in \mathbb{Z}$ and $-3.2 \notin \mathbb{Z}$.

Def. Two sets A and B are equal

$$A = B \iff \forall x \, (x \in A \iff x \in B).$$

There is only one set with no elements, the empty set that is denoted ϕ, 0 or $\{\ \}$ (or null set).

Example. We have two sets

$A = \{a, b, c\}$,

$B = \{b, a, c\}$.

$\Rightarrow \quad A = B$.

Def. A set A is a subset of a set B is such that every element of A is also an element of B. (A is contained in B) $A \subset B$.

$\forall a \in A \Rightarrow a \in B$.

Example.

$A = \{a, b, c\} \subset B = \{a, e, b, d, c\}$

Give proofs of the following theorems.

T1. $A \subset A$.

T2. $\emptyset \subset A$.

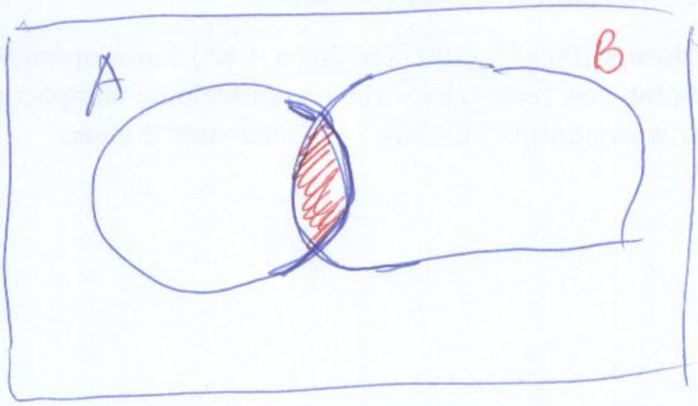
logical "and".

T3. $(A \subset B) \wedge (B \subset C) \Rightarrow A \subset C$.

T4. $(A \subset B) \wedge (B \subset A) \Leftrightarrow \boxed{A = B}$.

Def. The intersection of two sets A and B is a set denoted $A \cap B$ whose elements are those elements that belong to both A and B.

$$A \cap B = \{ x \mid x \in A \land x \in B \}$$



If S is a set (nonempty) of sets the intersection, denoted $\bigcap_{A \in S} A$, is the set whose elements are those elements that belong ~~to~~ to all sets in S.

$$\bigcap_{A \in S} = \{ x \mid (\forall A \in S)\ x \in A \}$$

Union.

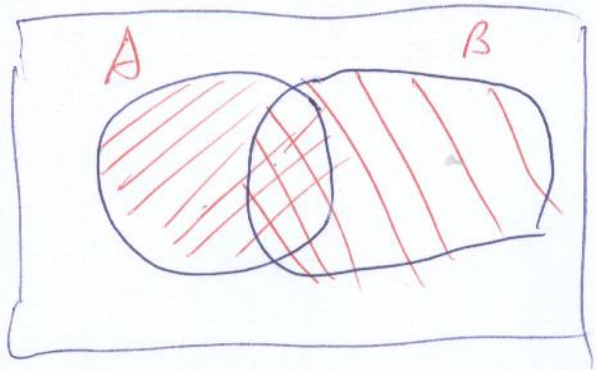
Def. The union of two sets A and B is a set denoted $\boxed{A \cup B}$ whose elements are those elements that belong to A or B or both.

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

(where $\vee$ denotes the logical "or").

If S is a set of sets, its union is a set

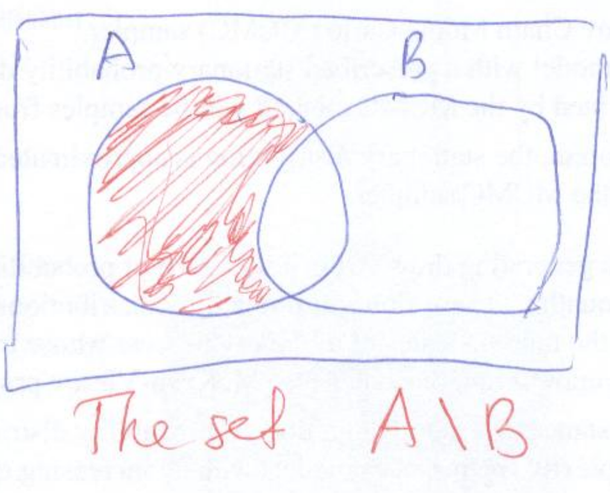
$$\bigcup_{A \in S} A = \{x \mid (\exists A \in S)\, x \in A\}.$$



Set difference

The set difference of two sets A and B, is a set, denoted $A \setminus B$ or $A - B$, whose elements are those elements that belong to A, but not to B:

$$A \setminus B = \{ x \mid x \in A \land x \notin B \}$$



The set $A \setminus B$

T. $A \setminus B = A \setminus (A \cap B)$

Example

$$\{1, 2, 3\} \setminus \{1, 3, 6, 8\} = \{2\}$$

$$\{1, 3, 6, 8\} \setminus \{1, 2, 3\} = ???$$

Relation

Def. A relation from a set A to a set B is a subset of $A \times B$.

Hence a relation R consists of orolered pairs (a, b), where $a \in A$ and $b \in B$.

If $(a, b) \in R$ we say that (a, b) is related to R. Note: aRb

Ex. 1. Universal relation

$$U_{A \times B} = \{(a, b) : \forall a \in A, \forall b \in B\}$$

Is a relationship where every element of a set A is related to every element of B.

If $U_{A \times A}$ we have that $(a, a) \in U_{A \times A}$

Ex 2.

$$R = \{ (a, b) \in \mathbb{R} \mid a < b \}.$$

hence $(a, b) \in R$ if and only if $a < b$.

Ex. 3.

$$A = \{1, 2, 3, 4, 5, 6\}, \quad B = \{1, 2, 3, 4\}.$$

Define $(a, b) \in R$ iff $a - b = 2k$

$\overline{(a-b) \bmod 2 = 0}$.

The domain of a relation $R \subseteq A \times B$ is defined as

$$\text{domain of } R = \{ a \in A \mid (a, b) \in R \text{ for some } b \in B \}.$$

The range is defined as

$$\text{range of } R = \{ b \in B \mid (a, b) \in R \text{ for some } a \in A \}.$$

Functions

A function is a special type of relation where each element of a set (the domain) is related to exactly one element of B set (the range).

Def. Relation $f \subset A \times B$ is a function if

$$[(x, y_1) \in f \wedge (x, y_2) \in f] \Rightarrow y_1 = y_2.$$

Notations:

$$x f y, \quad (x, y) \in f, \quad y = f(x).$$

<u>Injective</u> functions, are mappings between sets where each element in the domain (input) is associated with a unique element in the codomain (output, region).

Def $f : X \to Y$ is injective if for all x_1, x_2 in X, if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

Equivalently, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

$$\boxed{\mathcal{D}_f = X, \quad E_f \subset Y.}$$

Surjective function f is a function where every element in the codomain (the region, the set of possible output values) is mapped to by at least one element in the domain.

1) $D_f = X$, 2) $E_f = Y$

$$\left(\forall y \in Y \Rightarrow \exists \text{ at least one } x \in X : f(x) = y \right).$$

A bijective function (or one-to-one correspondence) is a function that is both injective ~~fone~~ and surjective.

For every element in the domain there is exactly one unique correspondence element in the codomain (region).

Examples.

1. $f_1 = (x, \sin x)$, $X = R$, $Y = [-1, 1]$

f_1 is not an injection, it is a surjection.

2. $f_2 = (x, \sin x)$, $X = [0, \frac{\pi}{2}]$, $Y = [-1, 1]$

f_2 is an injection, but it is not a surjection.

3. $f_3 = (x, \sin x)$, $X = [-\frac{\pi}{2}, \frac{\pi}{2}]$, $Y = [-1, 1]$

f_3 is a bijection

Equivalence relation

Def. Reflexive relation is a relation of elements of a set A $(A \times A)$ such that each element of the set is related to itself

A relation $S \subset A \times A$

$$\boxed{\forall a \in A \Rightarrow (a, a) \in S}$$

Def. Symmetric relation is a relation of elements of a set X $(S \subset X \times X)$ if

$$\forall a, b \in X : (a, b) \in S \Rightarrow (b, a) \in S$$

$$aSb \Leftrightarrow bSa.$$

Example. $S: (a, b) \in S$ if $a = b$.

Ex. 2 Relation "$a < b$" is not symmetric.

Def. A relation S on a set X is transitive if for all elements $a, b, c \in X$ whenever S relates a to b, and b to c, then S also relates a to c.

$\forall a, b, c \in X$: if aSb and bSc, then aSc.

Ex 3. Relation for real numbers $(a, b) \in S$ if $a < b$ is transitive

$a < b$ and $b < c \Rightarrow a < c$, i.e. $(a, c) \in S$.

Def. A relation S on a set X is equivalence relation if it is reflexive, symmetric and transitive.

Ex. Relation for real numbers $\mathbb{R}$ $(a, b) \in S$ if $a = b$, $a, b \in \mathbb{R}$ is an equivalence relation.